1) Let $F_{x,r} : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$F_{x,r}(z) = \|x - z\|^2 - r^2.$$

Note that when $r > 0$, 0 is a regular value of $F_{x,r}$ and $(F_{x,r})^{-1}(0)$ is the sphere of radius r, centered at x. Furthermore, $(F_{x_1,r_1})^{-1}(0) \pitchfork (F_{x_2,r_2})^{-1}(0)$ if and only if $(0,0)$ is a regular value of

$$G(z) = (F_{x_1,r_1}(z),\ F_{x_2,r_2}(z))$$

Note that

$$dG(z) = \begin{pmatrix} 2(z^{(1)} - x_1^{(1)}) & 2(z^{(2)} - x_1^{(2)}) & 2(z^{(3)} - x_1^{(3)}) \\ 2(z^{(1)} - x_2^{(1)}) & 2(z^{(2)} - x_2^{(2)}) & 2(z^{(3)} - x_2^{(3)}) \end{pmatrix}$$

This matrix has full rank if and only if its rows are linearly indendent. Hence, the intersection is non-transverse if and only if there exists $z \in \mathbb{R}^3$ belonging to the intersection such that

$$(*) \quad z - x_1 = \lambda(z - x_2)$$

for some $\lambda \in \mathbb{R}$. By taking norms, and using that $\|z - x_i\| = r_i$, we conclude that $|\lambda| = r_1/r_2$. Finally, equation $(*)$ implies that x_1, x_2 and z lie on the same line. It then follows that non-transversality occurs exactly when either:

$$\|x_1 - x_2\| = |r_1 - r_2| \quad \text{or} \quad \|x_1 - x_2\| = r_1 + r_2$$

2) Define $F : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$F(x,y) = (x, y, f(x,y))$$

Then F is a homeomorphism onto its image, and

$$dF_{(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x(x,y) & f_y(x,y) \end{pmatrix}$$

Since F is an immersion and homeomorphism onto its image, it is an embedding.

The tangent bundle to Γ_f is given by

(1) $\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ f_x(x,y) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ f_y(x,y) \end{pmatrix} \right\}$

and

(2) $\ker \varphi_{x,y}$ whre

$$\varphi_{x,y}(v_1, v_2, v_3) = v_3 - f_y(x,y)v_2 - f_x(x,y)v_1$$

Finally, if f and g are C^∞, note that Γ_f and Γ_{g+c} intersect in $f(h^{-1}(c))$, where $h(x,y) = g(x,y) - f(x,y)$. Furthermore, the planes in (1) coincide for f and g if and only if $\nabla f = \nabla g$, or $\nabla h = 0$.

Hence, we may find a transverse intersection when h has a regular value. By Sard's Theorem, the set of regular values has full measure, and by definition of h, the image is a nontrivial interval unless $f-g$ is a constant. Hence $\Gamma_f \pitchfork \Gamma_{g+c}$ for some c unless $\nabla f \equiv \nabla g$.

---

3) Consider the function $F(A,v) = Av$, so $F: M_2(\mathbb{R}) \times S^1 \to \mathbb{R}^2$. We claim that $F \pitchfork S^1$. Indeed,

$$F\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}v_1 \\ v_2\end{pmatrix}\right) = \begin{pmatrix}\alpha v_1 + \beta v_2 \\ \gamma v_1 + \delta v_2\end{pmatrix}$$

Therefore,

$$dF\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}v_1 \\ v_2\end{pmatrix}\right)\left(\begin{pmatrix}a & b \\ c & d\end{pmatrix}, \begin{pmatrix}w_1 \\ w_2\end{pmatrix}\right)$$

$$(*) \qquad = \begin{pmatrix}\alpha w_1 + a v_1 + \beta w_2 + b v_2 \\ \gamma w_1 + c v_2 + \gamma w_2 + d v_2\end{pmatrix}, \quad v_1 w_1 + v_2 w_2 = 0$$

Now, if $F(A,v) \in S^1$, then $v \neq 0$. Then by equation $(*)$, $dF(A,v)$ is onto, by choosing $\begin{pmatrix}a & b \\ c & d\end{pmatrix}$ appropriately, and taking $w = 0$. By Sard's Theorem, for almost every A, $F_A \pitchfork S^1$.

To see when the intersection is nontrivial, note that $F_A(v) = Av \in S^1$ if and only if $\|Av\| = 1$. Thus, we require that

$$\|A\| = \sup_{v \in S^1} \|Av\| \geq 1$$

and

$$\inf_{v \in S^1} \|Av\| \leq 1.$$

To see when F_A is an immersion, note that $T_v S^1 = \mathbb{R} \cdot \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$. Then

$$dF_A(v)\left(t\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}\right) = tA\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix},$$

and this in nonzero at every point if and only if $\ker A = 0$. That is, when A is invertible. Since $(F_A)^{-1} = F_{A^{-1}}$, it follows that ~~F_A~~ TFAE:

- F_A is an immersion
- F_A is an embedding
- A is invertible.

4) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and note that

$$F(A) = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

Therefore,

$$dF(A) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2a & b & c & 0 \\ c & a+d & 0 & c \\ b & 0 & a+d & b \\ 0 & b & c & 2d \end{pmatrix} \end{array}$$